Calculus I Brain Summary

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This is a list of all the thoughts located in the Calculus I Brain. Each thought is followed by a statement indicating the content associated by that thought. Italicized thoughts are all example problems. Thoughts followed by (jump) are jump thoughts, which are located on the left hand side of the central thought in the Brain matrix. Jump thoughts often take the user to entirely different sections of the Brain or provide reference information about the central thought.

Review of Basic Concepts – A short refresher course of concepts covered in Algebra, Trig and Geometry

- **Cartesian Plane** – A brief discussion about the Cartesian plan and how it works
  - **Definition of Circle in Plane** – Gives the standard equation of a circle
  - **Distance Formula** – Just what it says
  - **Midpoint Formula** – Just what it says
  - **Standard Form of Circle** – Same as "Definition of Circle in Plane"
- **Function Review** – Explanation of function notation and how a function is defined
  - **Function Representation** – Equivalent means of representation of functions
    - ***Examples*** – An example of how we utilize function representation
  - **Translation/Combination** – Shifting functions around the Cartesian plane
    - ***Composition Examples*** – Examples of composing two functions with each other
      - ***Even More Examples*** – 2 examples of real world applications
    - ***Trans/Combo Examples*** – Yet more examples involving shifting graphs of functions around the Cartesian Plane
    - ***Trig Example*** – A graph showing how to scale trig functions based on amplitude and frequency
  - **Types of Functions** – A brief summary of the different kinds of functions and properties of said functions
- **Geometry Review** – A brief reminder of the various formulas involving geometric shapes
  - **Related Rates (jump)** – This link takes you to the area of Related Rates in Differentiation,
  - **Circles** – Area, circumference of different shapes based on circles, such as ellipses, sectors, and rings
  - **Triangles** – Includes equilateral, right, and general triangle formulas
  - **Other** – Other common geometric shapes such as trapezoids and parallelograms
  - **Volumes** – Formulas of 3-dimensional shapes such as cones, spheres, and wedges
- **Real Numbers and Number Line** – Review of key concepts about real numbers, important to understanding calculus
  - **Absolute Value** – Definition and properties of the absolute value operator
  - **Definition of Order** – Basically a review of inequalities and intervals
  - **Intervals** – Intervals and their notation as used in mathematics
  - **Properties of Inequalities** – Just what it sounds like

Limits – An introduction to limits with an informal definition

- **Example Problems (jump)** – Several examples of limits using the informal definition
  - ***Example 1*** – Uses a technique to factor a denominator and cancel like terms
  - ***Example 2*** – A special trick is introduced to eliminate a hostile term to make the limit easier to evaluate
  - ***Example 3*** – Rationalization of the expression to get rid of radical signs
- **Limits at Infinity (jump)** – More fully covered under Differentiation | Advanced Principles
- **Basic Limits** – 3 simple formulas for evaluating limits of a particular form
  - **Functions vs. Limits** – A brief discussion of the difference between a function and a limit
- **Continuity and One-Sided Limits** – Definition of continuity (and discontinuity)
  - **Continuity on Closed Interval** – Definition of what it means to be continuous on a closed interval
  - **Existence of a Limit** – Definition of when a (two-sided) limit exists
    - ***Example*** – Shows an example of when a (two-sided) limit does not exist
  - **Intermediate Value Theorem** – A brief discussion of the IVT and how it works graphically
  - **One-Sided Limits** – Explanation of what they are, including a Flash Video Example
  - **Properties of Continuity** – Several common properties exploited by various functions
- **Delta-Epsilon Definition** – The formal definition of what a limit actually is, explained in graphical terms
  - ***Example 1*** – Shows how to use the D-E Definition, using a specified value of ε
  - ***Example 2*** – Shows how to *choose* a value for δ to use the D-E definition in order to prove that a given limit is true
  - ***Example 3*** – Uses a graph to find a particular value for δ
- **Infinite Limits** – Explanation of what happens when a given limit equals ∞
  - **Properties** – Several characteristic properties of what happens when limits equal ∞
  - **Vertical Asymptotes** – The graphical explanation of infinite limits
- **Properties of Limits** – Just what it sounds like
  - ***Example 1*** – Finding the limit using a trick of factoring, then canceling factors
  - ***Example 2*** – Finding the limit by rationalizing the numerator

Example 3 – Finding the limit using a trick of finding the least common denominator
Example 4 – Finding the limit of a function involving absolute values

Special Limits – A list of limits of different types of functions
Composite Functions – Limit of two functions composed with each other
Polynomial Functions – Limit of a polynomial function
Radical Functions – Limit of a function involving a $\sqrt{}$ symbol
Rational Functions – Limit of one polynomial function divided by another
Trig Functions – Limit of sine, cosine, tangent and their associated cofunctions

Squeeze Theorem – How to find the limit of a function through indirect means
Example – An application of the Squeeze Theorem

Tangent Line Problem – The problem which gave rise to the differentiation side of calculus
Derivative at a Point – Basic definition of what the derivative is in graphical terms including a Flash Video Example
Derivative as a Function – How to find the slope of a function at *any* point on the curve
Example – Uses the definition of the derivative to find the slope of a function
Derivative Notation – Various ways in which we can express the notion of a derivative
Graphing the Derivative – How to graph the derivative based on key points
Differentiability and Continuity – Any function that is differentiable is by definition also continuous
Tangent Line Examples
Example 1 – How to find the tangent line to a graph using a process of estimation
Example 2 – Uses the definition of the derivative at a point to find the tangent line to a curve
Example 3 – Finding the slope of a tangent line and a secant line connecting two points on a curve

Differentiation – One of the two elemental concepts in calculus, concerning the slope of a function at any point

Definitions (jump) – Basic concepts, mostly covered under the Tangent Line Problem (see above)

Basic Principles – Elementary rules and applications of differentiation
Applications – The two most basic applications are Related Rates and Rates of Change
Examples (jump) – 3 common examples of differentiation as used in physics and geometry
Circle – The rate of change of the area of an expanding circle
Optics – The rate of change of the focal length of a double-convex lens
Pressure-Volume – The rate of change of volume with respect to time as pressure increases
Rates of Change – One of the fundamental definitions of the derivative
Acceleration – The rate of change of velocity with respect to time
Velocity – The rate of change of position with respect to time
Average Velocity – The sort of velocity we usually deal with
Instantaneous Velocity – The calculus brand of velocity
Related Rates – Changing the rate of one variable with respect to another rate of change (Flash Video Example)
Chain Rule (jump) – A link to the Chain Rule, which is instrumental in solving this type of problem
Geometry Review (jump) – Another key element to solving Related Rates problems is understanding geometry
Method – Solving Related Rates (jump) – A brief summary of how to solve this type of problem
Example 1 – Find the change in height given a particular change in volume
Example 2 – Ideal Gas Law example finding change in volume given a change in pressure
Example 3 – Generic related rates example
Higher Order Derivatives – How to find the second, third and so on derivatives of a given function
More on Higher Order Derivatives – Further explanation on Higher Order Derivatives
Implicit Differentiation – A method of differentiating a function indirectly
Method of Implicit Diff (jump) – A summary of the technique of Implicit Differentiation
Example 1 – Polynomial function example
Example 2 – Trigonometric function example
Linear Approximations – Definition of the linearization of a function
Examples (jump)
Example 1 – Generic polynomial function example
Example 2 – Example involving the accuracy of the approximation

- **Differentials** – Definition of the differential and its graphical representation
  - **General Formulas (jump)** – Basic formulas involving differentials similar to those for derivatives
  - ***Example 1*** – Generic examples on how to find the differential of a given function
  - ***Example 2*** – Slightly more complicated example using differentials
  - **Relative Size** – Shows how big Δy is compared to dy as $\Delta x \to 0$
- **Rules** – The various elementary rules governing the derivative operator
  - **Rules Summary (jump)** – A list of all the rules for easy reference
  - ***Examples*** – Several examples using the rules of differentiation
    - ***Ex 1 – 4*** – Examples 1 through 4
      - ***Example 1***
      - ***Example 2***
      - ***Example 3***
      - ***Example 4***
    - ***Ex 5 – 7*** – Examples 5 through 7
      - ***Example 5***
      - ***Example 6***
      - ***Example 7***
  - **Chain Rule** – The cornerstone of differentiation and without which calculus would be impossible
    - **Related Rates (jump)** – Related Rates uses the Chain Rule extensively
    - ***Example 1*** – Simple example of the Chain Rule
    - ***Example 2*** – Much more complicated example involving the graphical shape known as the Witch of Maria Agnesi
  - **Constant Multiple Rule** – The constant can be factored out before differentiating the function
  - **Power Rules** – How to differentiate a function raised to a power
    - **General Power Rule** – Power Rule involving use of the Chain Rule
    - **Power Rule** – Simplest form of the General Power Rule, where u(x) = x
      - ***Example 4*** – Several examples showing how to use various rules of differentiation
  - **Product Rule** – How to differentiate two functions multiplied together
    - ***Product Rule Example*** – Several examples of how to apply the Product Rule
  - **Quotient Rule** – How to differentiate a rational function
    - ***Example 2*** – Same as Example 2 under Chain Rule
  - **Sum/Difference Rules** – How to differentiate the sum of two functions
    - ***Example 4*** – Same as Example 4 under Power Rule
  - **Trig Function Rules** – How to differentiate any of the six trig functions
    - **Sin/Cos** – Differentiation of sine and cosine
    - **Tan/Cot/Sec/Csc** – Differentiation of tangent, cotangent, secant, and cosecant

Advanced Principles – More advanced ideas about differentiation such as Extrema and Rolle's Theorem

- **Business and Econ** – Common formulas encountered in business and economic applications
- **Concavity** – Description of functions based on how much it curves
  - ***Example (jump)*** – A comprehensive example looking at the complete description of a function
  - **Points of Inflection** – The points on a curve at which the concavity changes from positive to negative and vice versa
  - **Second Derivative Test** – A test for finding points of relative extrema
  - **Test for Concavity** – How to determine if a function is concave up/down
- **Limits at Infinity** – What happens to the limit of a function as $x \to \infty$
  - **Limits (jump)** – A link to the basic concepts of limits
  - **Formal Definitions** – The Delta-Epsilon Definition as applied to limits at infinity
  - **Horizontal Asymptotes** – The graphical interpretation of limits at infinity (as opposed to infinite limits)
  - **Rational Limit at Infinity**
- **Min/Max Values** – The relative extrema of a functions, with Flash Video Example
  - ***Example 1 (jump)*** – How to use a graph to find relative extrema
  - ***Example 2 (jump)***
- **Extrema** – The high and low points on the graph of a function
  - ***Examples (jump)***
    - ***Example 1*** – Finding the absolute extrema of a function

Example 2 – Proving that a given function has no extrema
Guidelines (jump) – A brief summary of how to find the extrema of a function
Critical Numbers – The values used to determine extrema, since extrema can only occur at critical numbers
Extreme Value Theorem – On any given closed interval, there is at least one maximum and minimum
First Derivative Test – How to determine if a critical number is a maxima or minima
Increasing/Decreasing Functions – Basic definition
Guidelines (jump) – Summary of how to find intervals of increase and decrease
Test for Finding Inc/Dec Functions – Just what it sounds like
Relative Extrema – Basic definition
Newton's Method – How to approximate the roots of a function
Example 1 – Find the seventh root of 3 using Newton's Method
Example 2 – Find the indicated root of an equation to the desired accuracy
Optimization Problems – Applications of extrema
Example 1 – Fencing in a field using a fixed length of fence and maximizing the area enclosed
Example 2 – Find the specified volume of a box using the minimum amount of material
Rolle's Theorem – An existence theorem for critical points
Mean Value Theorem – Application of Rolle's Theorem
Cauchy MVT (jump) – Broader explanation of the Mean Value Theorem
Example – Application of the Mean Value Theorem involving the speed of a car

Integration – The other fundamental concept covered in calculus, concerning the area underneath a curve

Basic Integration Formulas (jump) – A summary of the most commonly used integration formulas
Guidelines for Integration (jump) – A set of steps for approaching problems involving integrals
Procedures (jump) – How to transform certain integrands to fit a particular integration formula
Basics – The basic concept of the integral and its associated properties
Antiderivatives – One interpretation of the indefinite integral
Basic Rules (jump) – A list of simple integration formulas
Examples
Example 1 – Finding the antiderivative of a polynomial function
Example 2 – Finding the linear density of a rod
Example 3 – Finding the height of a cliff by dropping a stone off of it
Integral Notation – The most common means of representing integrals in mathematics
Area – The problem for which the definite integral was developed to solve
Area of Region in Plane – How to find the area of a planar region
Limit of Lower/Upper Sums – A prelude to the infamous Riemann Sum
Example 1 – Using lower/upper sums to approximate the area under curve
Sigma Notation – Definition of what sigma notation actually means
Riemann Sums (jump) – A link to the Riemann Sum, which sets the stage for the Definite Integral
Summation Formulas – Common summation formulas
Summation Properties – Properties of summation along with proofs of those properties
Definite Integral – The area under a curve
Examples (jump)
Example 1 – Sketch a defined region, then find its area
Example 2 – Evaluate a Riemann Sum
Example 3 – Evaluate a Definite Integral
Area of a Region – What the Definite Integral is used for
Properties
Additive Interval Property – Separating one integral into two integrals
Basic Properties – Fundamental properties associated with the Definite Integral
Midpoint Rule of Integrals – A rule used for approximation of an area
Preservation of Inequality – Definite Integrals obey same laws of inequality as any real number
Riemann Sums – Formal explanation of Definite Integral

Sigma Notation (jump) – Link back to sigma notation for more clarification of Riemann Sums

Fundamental Theorem – The link between differentiation and integration

Examples (jump)

Example 1 – Evaluate several definite integrals

Example 2 – Find the area of a region

Example 3 – Find the area of a region

2nd Fundamental Theorem – The other link between differentiation and integration

Example 1 – Integrate a function using the 2nd Fundamental Theorem

Example 2 – Find the derivative of a function with the 2nd Fundamental Theorem

Example 3 – Find the derivative of a function

Example 4 – Find the derivative of a function

Average Value on an Interval – How to find the average value of a function over a particular interval

Example 1 – Sketch the graph of a function and find its average value

MVT for Integrals – The Mean Value Theorem as it applies to integrals

Example 1 – Find the mean value of an integral

Integration by Substitution – The most fundamental and widely used technique of integration

Guidelines (jump) – A summary of the technique

Change of Variables – Also known as *u*-substitution

CV for Definite Integrals – *u*-substitution as it applies to the Definite Integral

Example 1 – Evaluate a definite integral using u-substitution

Example 2 – Evaluate a definite integral using u-substitution

Even and Odd Functions – How to integrate even and odd functions

General Power Rule – The General Power Rule of Differentiation in reverse

Integration of Composite Function – What *u*-substitution is really all about

Numerical Integration – Approximation methods of integration

Error – The difference between the approximation and the actual value

Integrals of Quadratics – How to quickly integrate a 2nd degree polynomial

Simpson's Rule – Approximating an integral using 2nd degree polynomials

Trapezoidal Rule – Approximating an integral by adding up trapezoids instead of rectangles

Techniques – Various techniques for evaluating integrals

Improper Integrals – Integrals involving infinite limits of integration

Infinite Discontinuity – Place where the integrand equals ∞ for some value c in the interval [a, b]

Special Type – A special type of improper integral based on geometric series

Indeterminate Forms – Numbers involving 0 and ∞ that make it impossible to determine what the number actually is

L'Hôpital's Rule – Method to transform Indeterminate Forms into determinate forms

Integration by Parts – How to integrate two functions multiplied together

Guidelines (jump) – A quick summary of IBP

Common Integrals – A short list of integrals that can be solved by IBP relatively easily

Partial Fractions – A technique for integrating rational functions

Guidelines (jump) – A quick summary of how to apply Partial Fractions

Trig Substitution – Integration technique involving transforming integrands into products of powers of trig functions

Special Formulas – Commonly occurring integration formulas

Trigonometric Integrals – Integration technique involving products of powers of trig functions

Wallis's Formulas (jump) – Special formula for evaluating cos(x) raised to some power

Integrals w/ Sec/Tan – Integration of powers of secant and tangent functions

Integrals w/ Sin/Cos – Integration of powers of sine and cosine functions

Applications – Real world type applications of the Definite Integral

Arc Length – How arc length of a function is defined

Examples (jump)

Example 1 – Finding the length of an arc over a given interval

Area of Surface of Revolution – How to find the surface area of a function revolved about an axis

Surface of Revolution – What happens when a function is revolved about an axis

Area Between 2 Curves – How to find the area enclosed by two given functions
- ***Example 1*** – Find the area of a shaded region
- ***Example 2*** – Find the area of a shaded region shaped like a tie-fighter

Fluid Pressure and Force – The force per unit area exerted by a fluid such as water or air
- **Fluid Force** – The force exerted on a surface by a fluid such as water or air

Moments and Centers of Mass
- **2-D System Moments** – Expanding moments of a system from a line into a plane
- **Linear System Moments** – Point around which various masses cancel each other out to maintain a certain equlibrium
- **Planar Lamina** – A continuous flat sheet of uniform density
- **Theorem of Pappus** – How to find the volume of an object based on its centroid revolved around an axis

Volume of Revolution – What happens when a curve is revolved around an axis
- **Shell Method** – One of two methods used to find the volume of a solid of revolution
  - ***Example 1*** – Find the volume of a solid of revolution
  - ***Example 2*** – Find the volume of a solid of revolution
  - ***Example 3*** – Find the volume of a solid of revolution, except use both the Shell Method and the Washer Method
    - **Solution: Shell Method** – One of 2 ways to solve Example 3
    - **Solution: Washer Method** – One of 2 ways to solve Example 3
- **Washer Method** – The other of two methods used to find the volume of a solid of revolution
  - **Disc Method** – A special case of the Washer Method, where the inner radius of the washer equals zero
    - ***Example 1*** – Find the volume of a solid of revolution
    - ***Example 2*** – Find the volume of a solid of revolution
  - **Solids w/Known Cross Sections** – Quick summary of formulas for solids whose cross-sections are known
  - **Washer Method** – The full explanation of the Washer Method
    - ***Example 1*** – Find the volume of a solid of revolution
    - **Example *2*** – Find the volume of a solid of revolution
    - ***Example 3*** – Find the volume of a solid of revolution, but use both the Shell Method and the Washer Method
      - **Solution: Shell Method** – One of 2 ways to solve Example 3
      - **Solution: Washer Method** – One of 2 ways to solve Example 3

Work – One of the most common applications of calculus, involving a force moving an object over some distance
- **Work Done by a Variable Force** – The work done by a force that is changing at some given rate
  - ***Example (jump)***
    - ***Example 1*** – Find the work done by a cable lifting a car up a mine shaft
    - ***Example 2*** – Find the work required to empty a tank full of water
    - ***Example 3*** – Find the work required to empty a tank full of gasoline
  - **Coulomb's Law** – Work done by moving charged particles around
  - **Hooke's Law** – Work performed by springs
    - ***Example 1*** – Find the work required to stretch a spring
  - **Law of Universal Gravitation** – Work performed by any two masses (such as the Earth and the Moon)
    - ***Example 1*** – Find the work required to put a satellite in geostationary orbit